Homework II

1. Theorem: Suppose a and b are integers, p is prime and $p \mid ab$. Then $p \mid a$ or $p \mid b$.
Use this theorem to show that if p and q are distinct primes, then $\tau(pq) = 4$.

2. Show by induction that if a prime p divides a product of n numbers, then it divides at least one of the numbers.

3. Show that if the square of no prime divides b and $a^2 \mid bc^2$, then $a \mid c$.

4. Show that $a \mid b$ if and only if whenever p is prime and $p^k \mid a$, then $p^k \mid b$.

5. Suppose p is a prime and $k > 0$. We say that p^k divides a exactly and write $p^k \| a$, in case $p^k \mid a$ but $p^k \nmid a$.
Prove that if a and b are positive integers, p is prime, and $a + b = 2p - 1$, then $p \| a! \cdot b!$.

6. Show that a and b are relatively prime if and only if whenever p is prime and $p \mid a$, then $p \nmid b$.

7. Show by induction on n that for all positive integers n

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^n} > \frac{n+1}{2}$$

Conclude that $1 + \frac{1}{2} + \cdots + \frac{1}{n}$ can be made arbitrarily large.

8. Prove that there are infinitely many primes (such as 3, 7, 11 and 19) that are congruent to 3 (mod 4).

Hint: Suppose there are only $p_1, \ldots, p_k$. Show that $4p_1p_2\cdots p_k - 1$ must have a prime divisor $\equiv 3 \pmod 4$.

See the problem: How many primes are congruent to 0 (mod 4)? To 2 (mod 4)?

9. Prove that there are infinitely many primes congruent to 5 (mod 6).

10. Disprove the statement: If p_i is the ith prime, then $p_1p_2\cdots p_k + 1$ is prime for all positive integers k.